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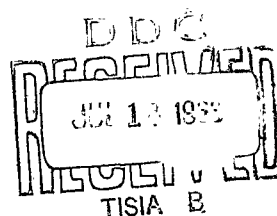
PROPAGATION OF WAVES IN INHOMOGENEOUS MAGNETO-ACTIVE  
PLASMA COLUMNS

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ABSTRACT

A preliminary theoretical investigation of the propagation of a quasi-static E mode in a plasma column with non-uniform electron density distribution in the cross-section, has already been performed in the special case of a density independent  $\epsilon_s/\epsilon_i$  ratio (Technical Summary Report N.2).

Starting from Maxwell's equations, a more general case is here considered, where a parabolic distribution has been taken as an analytical model for the electron density profile in the cross-section.

Electron density is supposed to vanish at a point outside the plasma column; in the limit for the zero point to go to infinity, uniform case equations are obtained.

Letting the metal walls move to infinity, or coincide with the plasma boundary, the completely filled waveguide and the plasma column in free space dispersion curves have been verified.

The Ritz variational method is used to minimize the error over the cross section when an approximation function for the field distribution in the plasma is assumed.

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## 1. Introduction

For analytical simplicity let us consider a plane geometry instead of a cylindrical one and let us suppose that a slab of inhomogeneous gyrotropic plasma is located between two parallel lossless metal plates, the distance between them being  $2D$  and the slab thickness being  $2d$ .

The coordinate reference frame is chosen so that the slab is perpendicular to the  $x$  axis with its surfaces at  $\pm d$ , and the  $z$  axis in the direction of propagating waves.

The magnetic field  $B$  points along the plates in the direction of the  $z$  axis. The plasma slab is inhomogeneous along the  $x$  axis and infinite in the  $y$  direction.

The field components and the electron density distribution function are expressed in terms of the independent variable  $u = \beta x, \beta$  being the longitudinal propagation constant.

The whole structure and the cross sectional distributions assumed for the  $E_z$  component and for the electron density are shown in fig. 1.

The analysis proceeds by calculating the longitudinal components of the electric field inside and outside the slab. Substituting these components into the boundary conditions then leads to a dispersion relation, whose solutions are given for the cases shown in fig. 2.

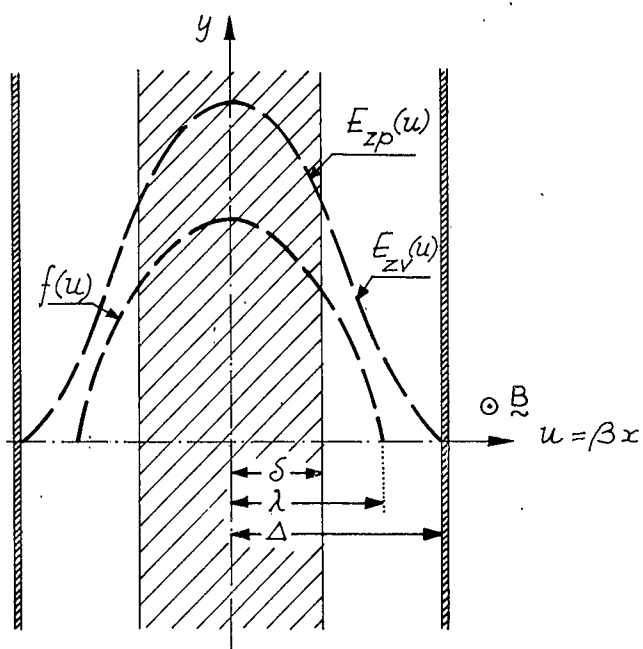


Fig. 1

The following symbols are used :

$E_{zp}(u)$  : cross sectional distribution of the longitudinal electric field inside the plasma.

$E_{zv}(u)$  : cross sectional distribution of the longitudinal electric field outside the plasma.

$E_0, E_1$  : amplitude constants of  $E_{zp}(u)$ .

$\beta$  : longitudinal propagation constant

$$u = \beta x$$

$2d$  : slab thickness

$2D$  : distance between plates.

$l$  : value of  $x$  at which the electron density is zero.

$$\delta = \beta d$$

$$\Delta = \beta D$$

$$\lambda = \beta l$$

$f(u)$  : cross sectional distribution of the electron density.

$L$  : second order differential linear operator.

$\mu$  : parameter characterizing the cross-sectional inhomogeneity.

$\omega_b$  : cyclotron angular frequency.

$\omega_p$  : plasma angular frequency at the center of the slab.

$$b_0 = \omega_p^2 / \omega^2$$

$$\alpha^2 = \omega_b^2 / \omega^2$$

$$\tau = 1 - (1 - \alpha^2) / b_0$$

$$\tau_0 = 1 - 1 / b_0$$

## 2. Derivation of the equation for $E_z(u)$ .

Starting from Maxwell's equations a wave equation for  $\underline{E} \equiv (E_x, E_y, E_z)$  is obtained :

$$\nabla^2 \underline{E} - \nabla \nabla \cdot \underline{E} + \omega^2 \mu_0 \tilde{\epsilon} \cdot \underline{E} = 0 \quad (1)$$

where  $\tilde{\epsilon}$  is the dyadic absolute permittivity.

If quasi-neutrality is assumed, (dielectric approximation) :

$$\nabla \cdot (\tilde{\epsilon} \cdot \underline{E}) = 0 \quad (2)$$

and, hence,  $\nabla \cdot \underline{E}$  is derived, the z component of eq.(1), in the quasi-static approximation, can be written in the form :

$$\frac{\partial^2 E_z}{\partial x^2} + \frac{1}{\epsilon_1} \frac{\partial \epsilon_1}{\partial x} \frac{\partial E_z}{\partial x} - \beta^2 \frac{\epsilon_3}{\epsilon_1} E_z = 0 \quad (3)$$

where  $\epsilon_1$  and  $\epsilon_3$  are the diagonal components of  $\tilde{\epsilon}$  and an  $\exp j(\omega t - \beta z)$  dependence is assumed.

In eq.(3) the second term is due to the inhomogeneity in the cross-section and the absence of any component of the magnetic field is due to the quasi-static approximation.

If the normalized variable  $u = \beta x$  is used eq.(3) can be written :

$$\frac{\partial^2 E_z}{\partial u^2} + \frac{1}{\epsilon_1} \frac{\partial \epsilon_1}{\partial u} \frac{\partial E_z}{\partial u} - \frac{\epsilon_3}{\epsilon_1} E_z = 0 \quad (4)$$



The following distributions in the cross section (plasma region) are now assumed :

$$\epsilon_1(u) = 1 - \frac{b(u)}{1 - \alpha^2}$$

$$\epsilon_3(u) = 1 - b(u)$$

with  $\alpha^2 = \frac{\omega_b^2}{\omega^2}$  and  $b(u) = b_0 f(u) = (\omega_p^2 / \omega^2) f(u)$ , where  $f(u)$  specifies a particular kind of density distribution.

If

$$f(u) = 1 - \frac{u^2}{\lambda^2}$$

where

$$\lambda^2 = \beta^2 l^2$$

$\epsilon_1(u)$  and  $\epsilon_3(u)$  can be rewritten

$$\epsilon_1(u) = 1 - \frac{b_0(\lambda^2 - u^2)}{\lambda^2(1 - \alpha^2)}$$

$$\epsilon_3(u) = 1 - b_0 \left(1 - \frac{u^2}{\lambda^2}\right)$$

Substituting these values into eq.(4) the following equation is obtained :

$$L(E_z) = (u^2 - \lambda^2 \tau) \frac{\partial^2 E_z}{\partial u^2} + 2u \frac{\partial E_z}{\partial u} + (\lambda^2 \tau_0 - u^2)(1 - \alpha^2) E_z = 0 \quad (5)$$

where

$$\tau = 1 - \frac{1 - \alpha^2}{b_0}$$

$$\tau_0 = 1 - \frac{1}{b_0}$$

and the first member of the equation has been designated with the operational symbol  $L(E_z)$ .

### 3. Solution of the equation for $E_z(u)$ inside the plasma.

A solution of eq. (5) in the plasma region can be obtained with the aid of the Ritz method by assuming a test function as an admissible analytical expression for  $E_z(u)$ .

If a linear combination of suitable functions is chosen for  $E_{zp}(u)$ ,

$$E_{zp}(u) = E_0 + \sum_k \alpha_k E_k(u)$$

a first order approximation is given by

$$E_{zp}(u) = E_0 + E_1 (\delta^2 - u^2) \quad (6)$$

where

$$\delta = \beta d$$

Introducing the expression (6) of  $E_{zp}(u)$  in eq. (5) the problem is reduced to a "minimum" problem with the following boundary conditions :

$$\begin{aligned} E_{zp}(\delta) &= E_{zv}(\delta) \\ \varepsilon_1(\delta) E_{xp}(\delta) &= E_{xv}(\delta) \\ E_{zv}(\Delta) &= 0 \end{aligned} \quad (7)$$

The problem can be mathematically formulated as follows:

$$E_1 \int_0^\delta (\delta^2 - u^2) L(\delta^2 - u^2) du = -E_0 \int_0^\delta (\delta^2 - u^2) \left( \lambda^2 \varepsilon_1(u) (1 - \alpha^2) \right) du \quad (8)$$

and if the degree of homogeneity in the cross-section is specified by a characteristic parameter  $\mu$  defined as follows :

$$\mu = \frac{\lambda}{\delta}$$

eq.(8) can be reduced to :

$$E_1 \left[ \delta^5 \left( \frac{2}{3} \tau \mu^2 - \frac{2}{5} \right) + (1 - \alpha^2) \left( \frac{4}{15} \mu^2 \tau_0 - \frac{4}{105} \right) \delta^7 \right] =$$

$$= - E_0 \delta^5 (1 - \alpha^2) \left( \frac{\mu^2 \tau_0}{3} - \frac{1}{15} \right) \quad (9)$$

#### 4. Solution of equation for $E_z(u)$ outside the plasma.

Outside the plasma eq.(5) reduces to the following ordinary second order differential equation :

$$\frac{\partial^2 E_z}{\partial u^2} - E_z = 0 \quad (10)$$

By imposing the boundary condition at  $u = \Delta$ , the solution of eq. (10) is :

$$E_{zv}(u) = A \left( e^{-u} - e^{-2\Delta+u} \right) \quad (11)$$

where  $A$  is an arbitrary amplitude constant.

#### 5. Derivation of the dispersion relation.

The eigen value  $\delta$  can be calculated by imposing the boundary conditions (7), at  $u = \delta$ .

When this is done, the following two equations are obtained :

$$E_0 = A \left( e^{-\delta} - e^{-2\Delta+\delta} \right) \quad (12)$$

$$E_1 = \frac{A}{2\delta \xi_1(\delta)} \left( e^{-\delta} + e^{-2\Delta+\delta} \right)$$

Substituting these expressions into eq.(9), we have :

$$\left(\frac{\tau\mu^2}{3} - \frac{1}{5}\right) + (1-\alpha^2)\left(\frac{2}{15}\tau_0\mu^2 - \frac{2}{105}\right)\delta^2 =$$

$$= -\delta \ln(\Delta-\delta)(1-\alpha^2)\left(\frac{\mu^2\tau_0}{3} - \frac{1}{15}\right)\left[1 - \frac{b_0(\mu^2-1)}{\mu^2(1-\alpha^2)}\right]^{(13)}$$

This is the dispersion relation in the general case of  $\delta < \lambda < \infty$  and  $\delta < \Delta < \infty$ . Some particular cases of special interest can be derived from this; they are shown in the following table.


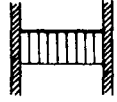


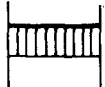
PHYSICAL STRUCTURE	DENSITY DISTRIBUTION	CURVE DESIGNATION
a) $\Delta = \delta$ (COMPLETELY FILLED WAVEGUIDE)	$\mu = 1$ (PARABOLIC)	$a_1$ 
	$\mu = \infty$ (UNIFORM)	$a_\infty$ 
b) $\Delta \rightarrow \infty$ (SLAB IN FREE SPACE)	$\mu = 1$ (PARABOLIC)	$b_1$ 
	$1 < \mu < \infty$ (PARABOLIC) (FOR EX. $\mu^2 = 2$ )	$b_2$ 
	$\mu = \infty$ (UNIFORM)	$b_\infty$ 

fig. 2

6. Particular cases.

In the a) case if we insert  $\Delta = \delta$  in eq. 13) the R.H.S vanishes and the following equation is left :

$$\left(\frac{1}{3}\mu^2 - \frac{1}{5}\right) + (1 - \alpha^2) \left(\frac{2}{15}T_0\mu^2 - \frac{2}{105}\right)\delta^2 = 0 \quad (14)$$

or the alternative form :

$$\left[\frac{\mu^2}{3} \left(1 - \frac{1 - \alpha^2}{b_0}\right) - \frac{1}{5}\right] + (1 - \alpha^2) \left[\frac{2\mu^2}{15} \left(1 - \frac{1}{b_0}\right) - \frac{2}{105}\right]\delta^2 = 0 \quad (15)$$

If  $\mu=1$  the above equation reduces to

$$\delta^2 = -\frac{7/6}{1 - \alpha^2} \frac{b_0 - 5/2(1 - \alpha^2)}{b_0 - 7/6} \quad (16)$$

which is the dispersion equation of a completely filled waveguide with a parabolic electron density distribution in the cross-section.

A plot of eq. (16) in the plane  $\bar{\omega}^2 = \frac{\omega^2}{\omega_p^2} = \frac{1}{b_0}$  vs.  $|\delta|$  yields the curves designated by  $a_1$ . (Each curve has been calculated for a fixed value of the parameter  $Q = \frac{\omega_b^2}{\omega_p^2} = \frac{\alpha^2}{b_0}$  (see tables I and I').

If  $\mu=\infty$  eq. (15) becomes :

$$\frac{1}{3} \left(1 - \frac{1 - \alpha^2}{b_0}\right) + (1 - \alpha^2) \left[\frac{2}{15} \left(1 - \frac{1}{b_0}\right)\right] = 0 \quad (17)$$

If  $\alpha=0$  (isotropic plasma)

$$\delta^2 = -\frac{5}{2}$$

there is no propagation.

If  $\alpha \neq 0$  (gyrotropic plasma)

$$\delta^2 = -\frac{5/2}{1-\alpha^2} \frac{b_0(1-\alpha^2)}{b_0-1} \quad (18)$$

which is the dispersion relation of a completely filled waveguide with a uniform distribution of the electron density in the cross-section.

If eq. (18) is plotted in the  $\bar{\omega}^2 - |\delta|$  plane, the curves  $a_\infty$  are obtained. (Tables I, I').

In the b) case if we let  $\Delta \rightarrow \infty$  in eq. (13), this eq. reduces to<sup>+</sup>:

$$\begin{aligned} & \left( \frac{\tau \mu^2}{3} - \frac{1}{5} \right) + (1-\alpha^2) \left( \frac{2\tau_0 \mu^2}{15} - \frac{2}{105} \right) \delta^2 = \\ & = -|\delta| (1-\alpha^2) \left( \frac{\tau_0 \mu^2}{3} - \frac{1}{15} \right) \left[ 1 - \frac{b_0(\mu^2-1)}{\mu^2(\alpha^2-1)} \right] \end{aligned} \quad (19)$$

which in terms of the normalized  $\bar{\omega}^2$ , has the form;

- 
- + When  $\Delta$  tends to infinity our physical structure reduces to a slab of plasma placed in an unlimited space. In this case  $\text{th}(\Delta - \delta) = 1$  and the absolute value of  $\delta$  must be considered, if reciprocity is to be respected.

From a mathematical stand point it is to be noticed that when  $\beta < 0$  the limit

$$\lim_{\Pi \rightarrow \infty} \beta d \text{th} \beta (\Pi - d)$$

tends to  $|\beta|d$  in the same manner as when  $\beta > 0$ .

For  $Q = \frac{1}{2}$

$$\left[ \left( \frac{2\mu^2}{15} - \frac{2}{105} \right) - \frac{2\mu^2 \bar{\omega}^2}{15} \right] \delta^2 + \left[ \frac{\mu^2}{3} - \frac{1}{15} - \frac{\mu^2 \bar{\omega}^2}{3} \right] \left[ 1 - \frac{\mu^2 - 1}{\mu^2 (\bar{\omega}^2 - \frac{1}{2})} \right] |\delta| + 2\bar{\omega}^2 \left[ \frac{\mu^2/3 - 1/5}{2\bar{\omega}^2 - 1} - \frac{\mu^2}{6} \right] = 0 \quad (20)$$

For  $Q = 2$

$$\left[ \frac{2\mu^2}{15} - \frac{2}{105} - \frac{2\mu^2 \bar{\omega}^2}{15} \right] \delta^2 + \left[ \frac{\mu^2}{3} - \frac{1}{15} - \frac{\mu^2 \bar{\omega}^2}{3} \right] \left[ 1 - \frac{\mu^2 - 1}{\mu^2 (\bar{\omega}^2 - 2)} \right] |\delta| + \bar{\omega}^2 \left[ \frac{\mu^2/3 - 1/5}{\bar{\omega}^2 - 2} - \frac{\mu^2}{3} \right] = 0 \quad (20')$$

If  $\mu^2 = 1$  eq. 20, 20' reduce to :

$$\left( \frac{12}{105} - \frac{2}{15} \bar{\omega}^2 \right) \delta^2 + \left( \frac{4}{15} - \frac{1}{3} \bar{\omega}^2 \right) |\delta| + \bar{\omega}^2 \left( \frac{4/15}{2\bar{\omega}^2 - 1} - \frac{1}{3} \right) = 0 \quad (21)$$

$$\delta^2 \left( \frac{12}{35} - \frac{2}{5} \bar{\omega}^2 \right) + \left( \frac{4}{5} - \bar{\omega}^2 \right) |\delta| + \bar{\omega}^2 \left( \frac{2}{5(\bar{\omega}^2 - 2)} - 1 \right) = 0 \quad (21')$$

Eqs. (21) are the dispersion relations for a slab of plasma in free space with a parabolic electron density distribution in the cross section. (curves b, tables II, II').

If  $\mu = \infty$ , eqs 20 reduce to :

$$\frac{2}{5} (1 - \bar{\omega}^2) \delta^2 + (1 - \bar{\omega}^2) \frac{2\bar{\omega}^2 - 3}{2\bar{\omega}^2 - 1} |\delta| + \bar{\omega}^2 \frac{3 - 2\bar{\omega}^2}{2\bar{\omega}^2 - 1} = 0 \quad (22)$$

$$\frac{2}{5} (1 - \bar{\omega}^2) \delta^2 + (1 - \bar{\omega}^2) \frac{\bar{\omega}^2 - 3}{\bar{\omega}^2 - 2} |\delta| + \bar{\omega}^2 \frac{3 - \bar{\omega}^2}{\bar{\omega}^2 - 2} = 0 \quad (22')$$

which are the dispersion relations for a slab of plasma in free space with a uniform electron density distribution in the cross-section. (Curves  $b_{\infty}$ ; tables II, II'). Dispersion curves, for the case  $\mu^2 = 2$  have also been calculated and plotted (curves  $b_2$ ).

7. Comparison between the approximate solution and the exact solution.

A direct analysis of the admissible propagation modes in a slab of uniform plasma in free space shows that a cosinus distribution of  $E_z$  in the cross-section is possible when a quasi static E mode is considered.

Imposition of boundary conditions yields a characteristic equation and a dispersion relation which is

$$\varepsilon_1 \sqrt{-\frac{\varepsilon_3}{\varepsilon_1}} \operatorname{tg} \left( \sqrt{-\frac{\varepsilon_3}{\varepsilon_1}} |\delta| \right) = 1 \quad (23)$$

where the modulus of  $|\delta|$  has been chosen in accordance with reciprocity principle.

In terms of the new normalized variable  $\bar{\omega}^2$  eq. (23) can be rewritten :

$$\operatorname{tg} \left[ |\delta| \sqrt{\frac{(1-\bar{\omega}^2)(\bar{\omega}^2-2)}{\bar{\omega}^2(\bar{\omega}^2-3)}} \right] = \frac{\bar{\omega}^2-2}{\bar{\omega}^2-3} \left[ \sqrt{\frac{\bar{\omega}^2(\bar{\omega}^2-3)}{(1-\bar{\omega}^2)(\bar{\omega}^2-2)}} \right] \quad (24)$$

This equation can be solved numerically and dispersion curves can be derived and compared with curve  $b_{\infty}$ . (Table IV, IV'). (A subscript "e" is used to indicate "exact solution" curves).



### CONCLUSIONS

Application of a variational method to a quasi-static problem, has been made. The choice of the test function, based on the knowledge of the field distribution in the uniform plasma case, represents a first order mathematical approximation capable of describing a mode with the lowest order transversal distribution.

An inspection of the dispersion curves, shows that, within the above limits :

1. - passing from cases  $a_0$  and  $b_0$  (uniform electron density distribution) to cases  $a_1$  and  $b_1$  (parabolic electron density distribution) the upper band shrinks.
2. - For the  $a_1$  curve, when the parameter  $a = \left(\frac{\omega_b}{\omega_p}\right)^2$  equals the critical value  $a_c = 0,456$  the upper band shrinks to zero and if  $a < a_c$ , propagation occurs only in the pass band  $0 < \omega^2 < a$

For the  $b$  curves (slab of plasma in free space) critical values of  $a$  have been computed for different  $\mu^2$  and the results plotted in fig. 3.

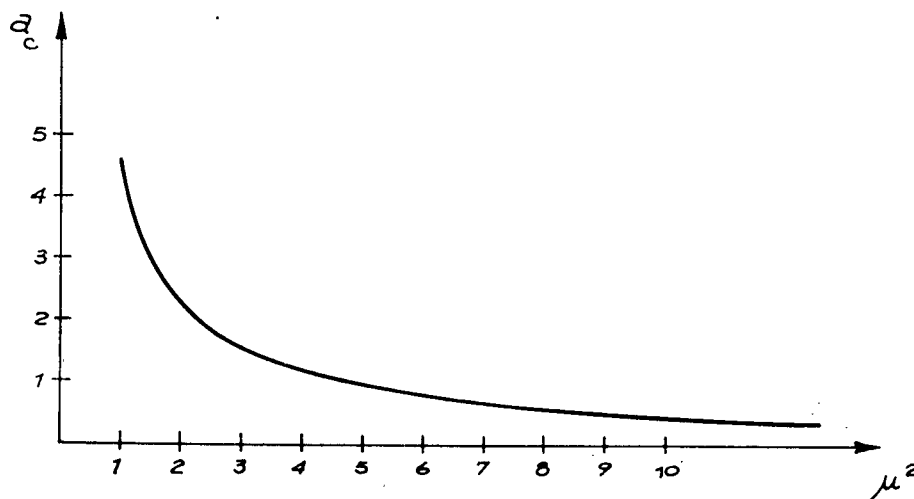


fig. 3

3. - The asymptotic  $\bar{\omega}^2$  value, equal to 1 in the uniform case, in the passage from  $a_\infty$  to  $a_1$  case, decreases whereas the asymptotic value equal to  $a$  remains unchanged. This is expected, since the average electron density, in the passage from  $a_\infty$  to  $a_1$  case, decreases as well.
4. - Letting the metal walls move away from the plasma boundary does not affect the behaviour of the dispersion curves as much as varying the electron distribution in the cross-section. (Table III and III').
5. - A comparative exam of the dispersion curves obtained by variational methods and by exact mathematical analysis, for a slab of plasma in free space, shows a fairly good agreement in the upper band, and an excellent agreement in the lower band when  $\omega_b > \omega_p$  (Table IV'). When  $\omega_b < \omega_p$  the lower branches of the curves show a different behaviour. (Table IV).

It has, in fact, been demonstrated that, with or without quasi static approximation, (see ref. 5) the lowest order mode dispersion curve approaches the asymptotic value  $\sqrt{\frac{\omega_p^2 + \omega_b^2}{2}}$  whenever  $\omega_p > \omega_b$  whereas application of the variational method provides only an asymptotic value for  $\omega = \omega_b$ .

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$$\frac{\omega_b^2}{\omega_p^2} = 0.5$$

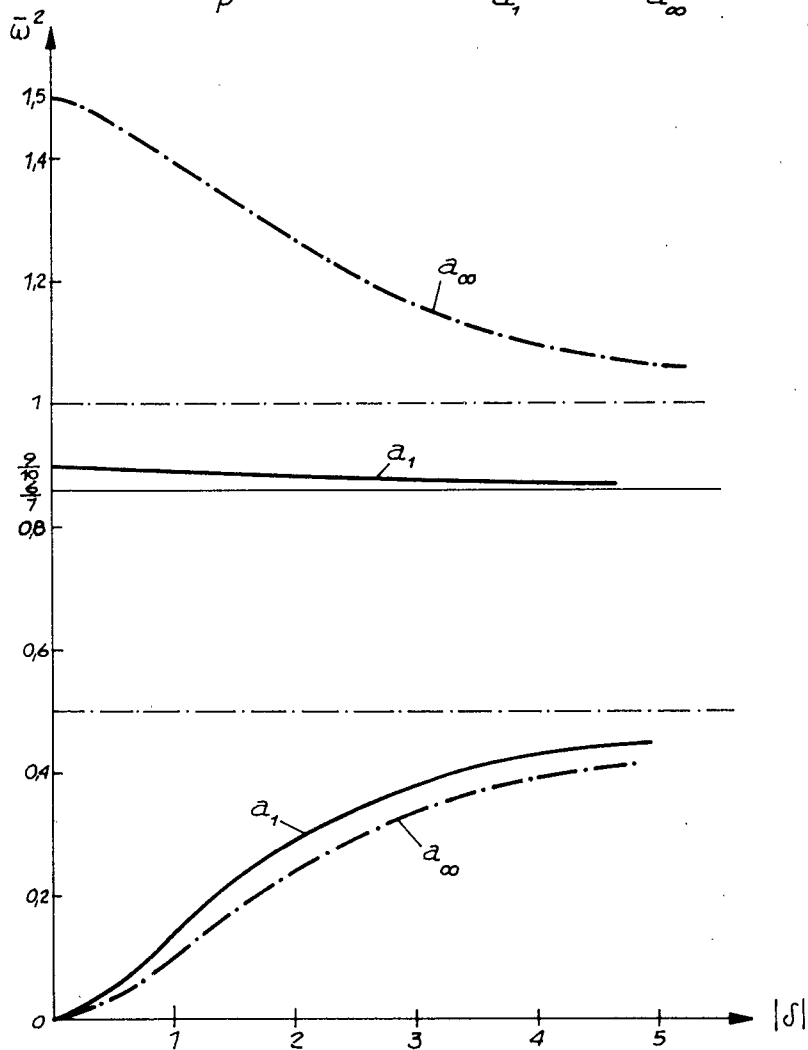


TABLE I - CURVES  $a_1, a_\infty$

$$\frac{\omega_b^2}{\omega_p^2} = 2$$

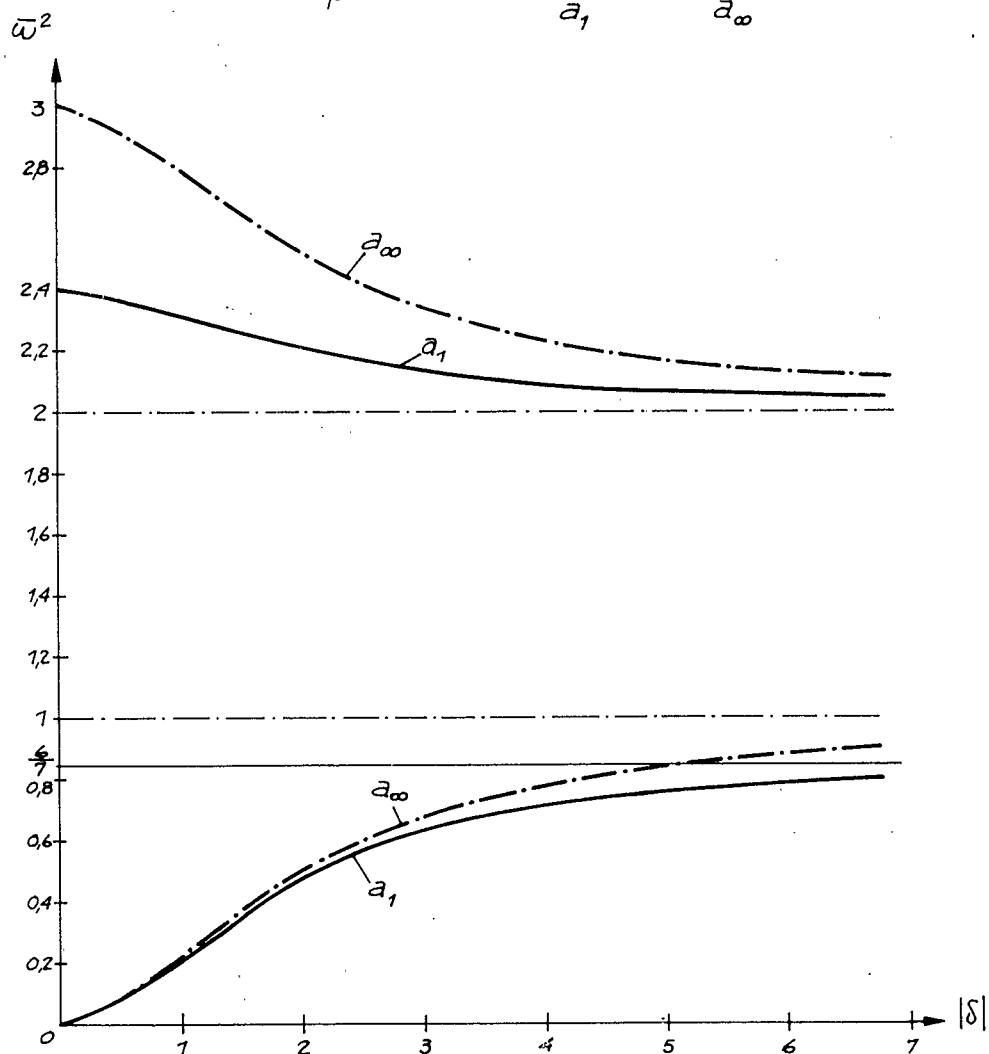


TABLE I' - CURVES  $a_1, a_\infty$

$$\frac{\omega_b^2}{\omega_p^2} = \frac{1}{2}$$

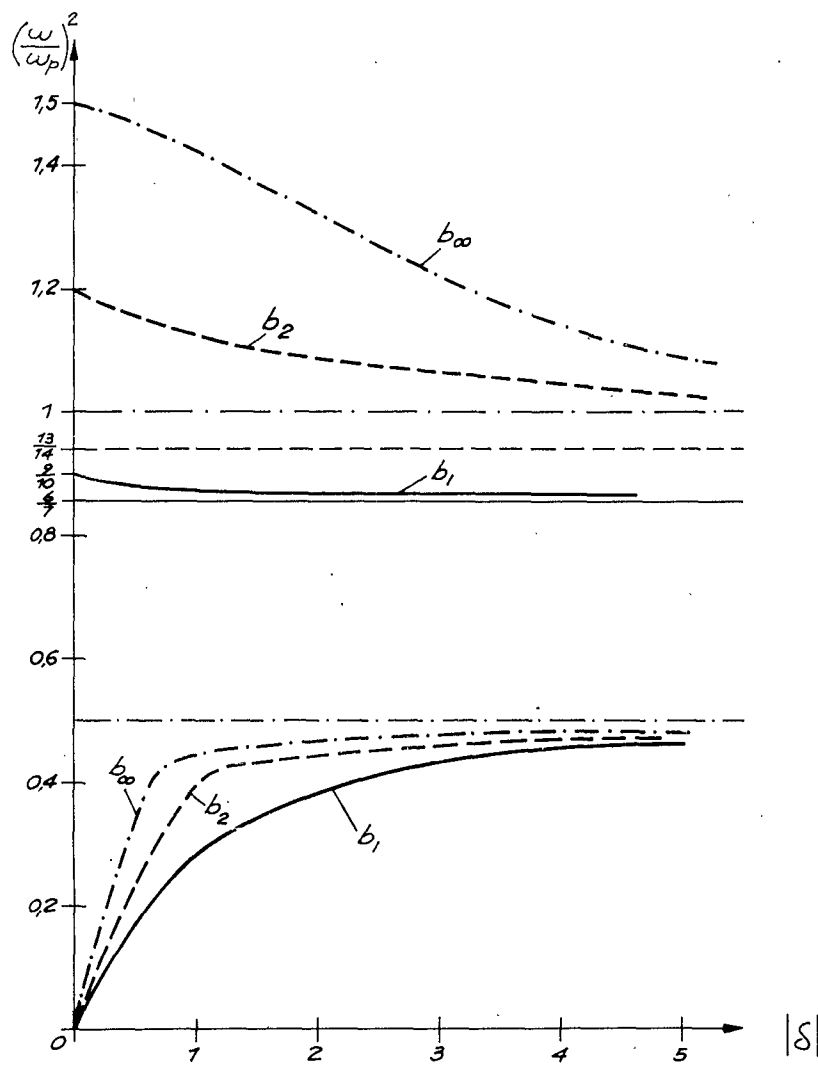
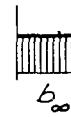


TABLE II - CURVES  $b_1, b_2, b_\infty$

$$\frac{\omega_b^2}{\omega_p^2} = 2$$

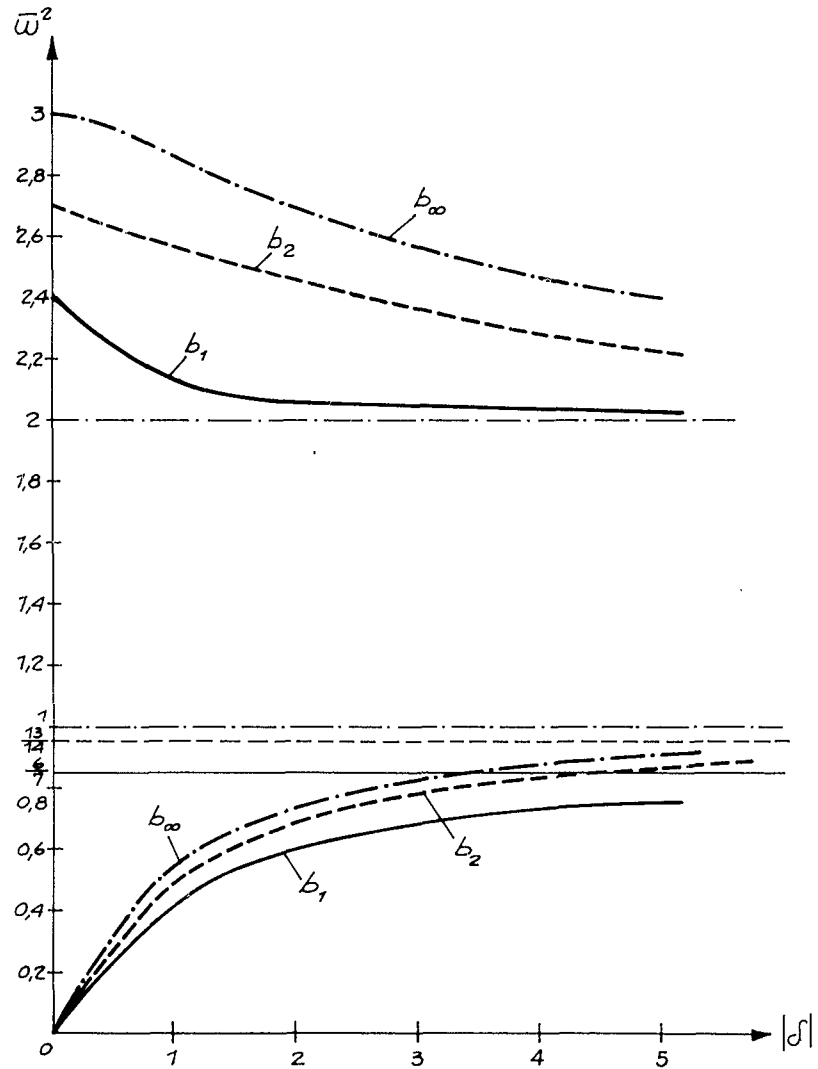
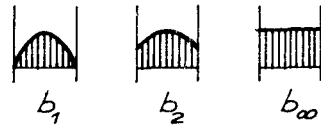


TABLE II' - CURVES  $b_1, b_2, b_\infty$

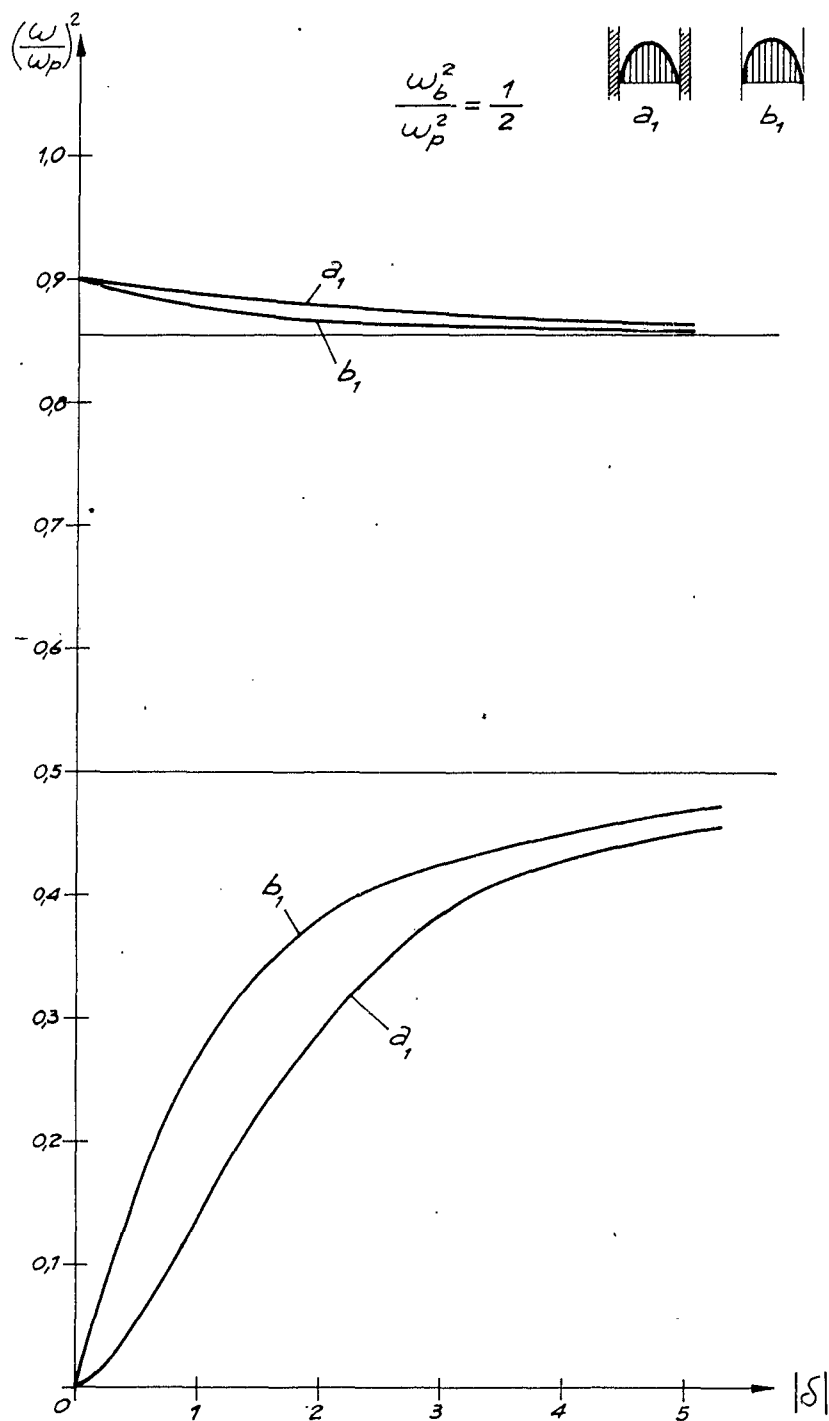


TABLE III - CURVES  $a_1, b_1$



$$\frac{\omega_b^2}{\omega_p^2} = 2$$

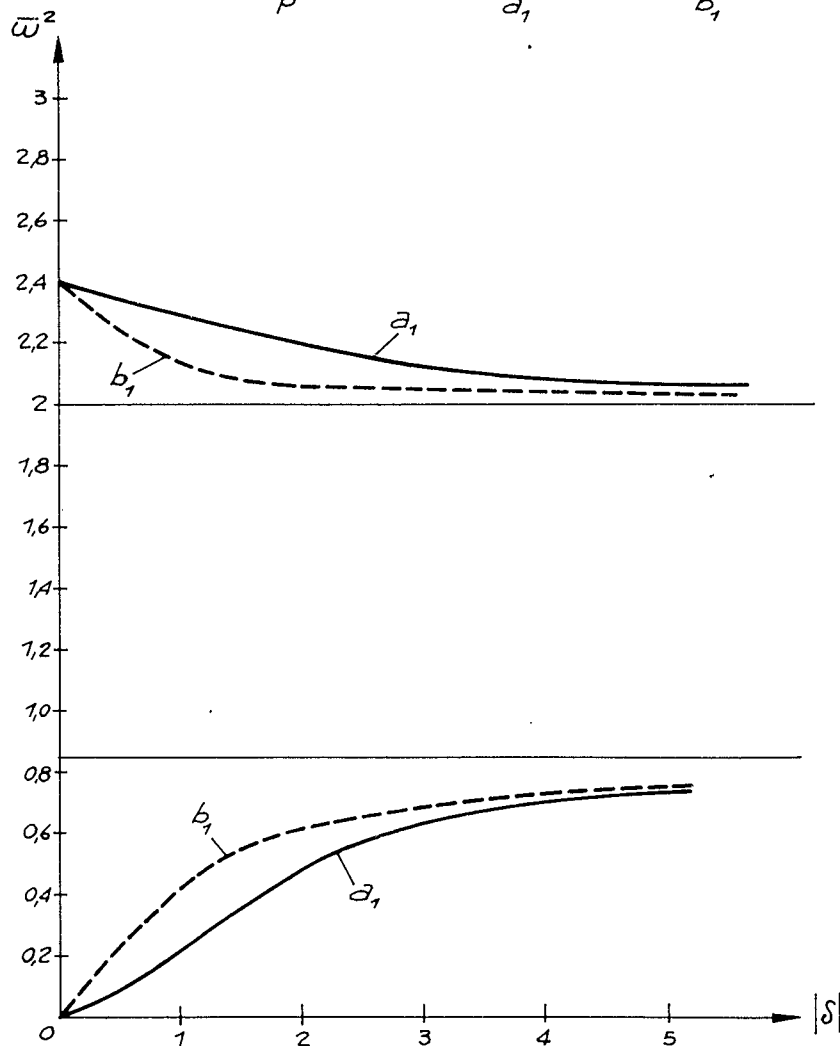


TABLE III' - CURVES  $a_1, b_1$

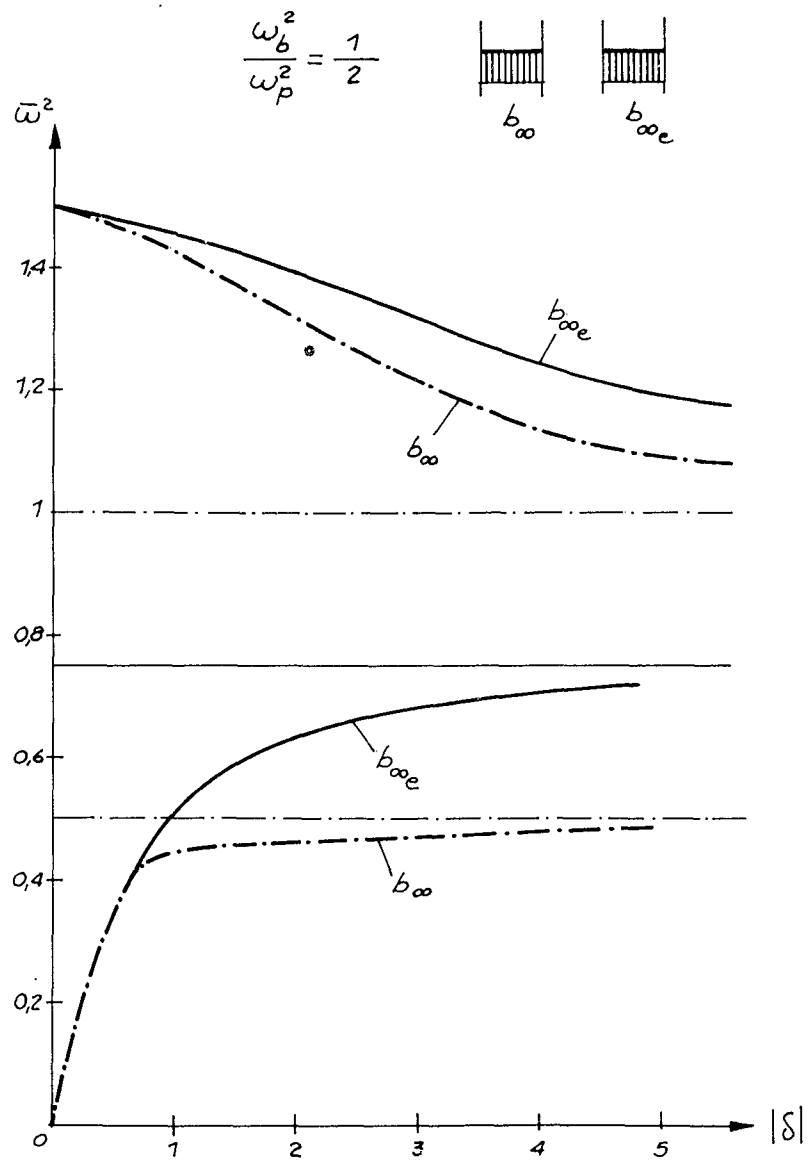


TABLE IV - CURVES  $b_{\infty}$ ,  $b_{\infty e}$

$$\frac{\omega_b^2}{\omega_p^2} = 2$$

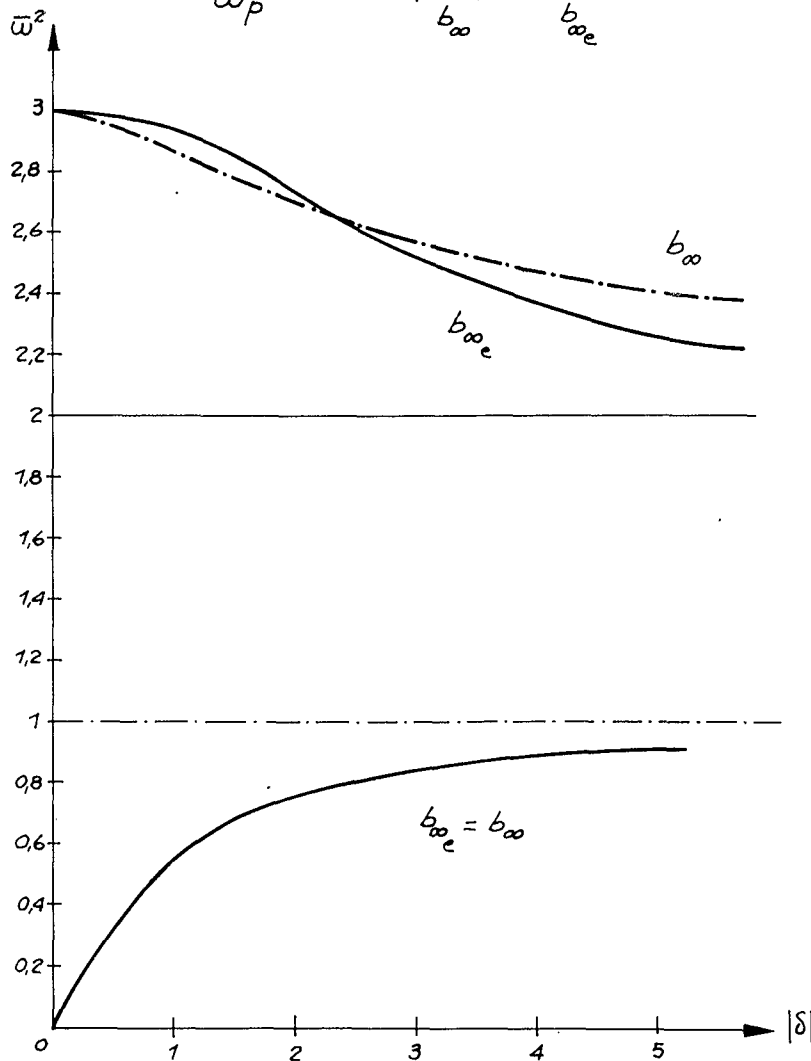


TABLE IV' - CURVES  $b_\infty, b_{\infty e}$